

# Singular linear statistics of the Laguerre Unitary Ensemble and Painlevé III: Double scaling analysis.

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## Abstract

We continue with the study of the Hankel determinant, defined by,  
 $D_n(t, \alpha) = \det \left( \int_0^\infty x^{j+k} w(x; t, \alpha) dx \right)_{j,k=0}^{n-1}$ , generated by a singularly perturbed Laguerre weight,  $w(x; t, \alpha) = x^\alpha e^{-x} e^{-t/x}$ ,  $x \in \mathbb{R}^+$ ,  $\alpha > 0$ ,  $t > 0$ , obtained through a deformation of the Laguerre weight function,  $w(x; 0, \alpha) = x^\alpha e^{-x}$ ,  $x \in \mathbb{R}^+$ ,  $\alpha > 0$ , via the multiplicative factor  $e^{-t/x}$ .

An earlier investigation was made on the finite  $n$  aspect of such determinants, which appeared in [20]. It was found that the logarithm of the Hankel determinant has an integral representation in terms of a particular Painlevé III ( $P_{III}$ , for short) and its  $t$  derivatives. In this paper we show that, under a double scaling, where  $n$ , the order of the Hankel matrix tends to  $\infty$ , and  $t$ , tends to  $0^+$ , the scaled—and therefore, in some sense, infinite dimensional—Hankel determinant, has an integral representation in terms of a  $C$  potential. The second order non-linear ode satisfied by  $C$ , after a change of variable, is another  $P_{III}$  transcendent, albeit with fewer number of parameters.

Expansions of the double scaled determinant for small and large parameter are obtained.

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# 1 Introduction.

The analysis of Hankel determinants plays an important role in random matrix theory [37]. The second author and his collaborators made use of a theorem in linear statistics to study Hankel determinants and the associated orthogonal polynomials, see [4, 6, 7, 22, 24] based on Dyson's Coulomb Fluid [31]. See also [17, 23]. Bonan, Nevai and others studied orthogonal polynomials with ladder operators. Works on the Hankel determinants generated by the deformations of classical weight, maybe found in [2, 8, 11, 12, 13, 14], and also in [5, 15, 18, 19, 26, 27].

Adler and Van Moerbeke obtained differential equations governing the logarithmic derivatives of Hankel determinants, and their generalization, using a multi-time approach, [1]. The connections between the Hankel determinants generated by quite general semi-classical weights are established in [9] and [10]. The asymptotics of  $n$  dimensional Toeplitz determinants with Fisher-Hartwig singularities are studied in [29], and in a review, [34].

The joint probability density function of the eigenvalues,  $\{x_j : j = 1, \dots, n\}$  for an ensemble of  $n \times n$  Hermitian matrix is given by [37]

$$p(x_1, x_2, \dots, x_n) \prod_{j=1}^n dx_j = \frac{1}{D_n[w]} \frac{1}{n!} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^n w(x_\ell) dx_\ell,$$

where  $D_n[w]$  is the normalization constant, or the partition function,

$$D_n[w] = \frac{1}{n!} \int_{\mathbb{R}_+^n} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^n w(x_\ell) dx_\ell.$$

Here  $w(x)$  is a positive weight function supported on  $\mathbb{R}_+$ , with moments given by,

$$\mu_j[w] := \int_{\mathbb{R}_+} x^j w(x) dx, \quad j \in \{0, 1, 2, \dots\}.$$

It is a well-known fact that  $D_n[w] = \det(\mu_{j+k}[w])_{0 \leq j, k \leq n-1}$ , and this gives a link between the determinant of the Hankel matrix  $(\mu_{j+k})$  and the partition function  $D_n[w]$  given by the multiple integral.

A linear statistics is defined to be the sum of a function  $f(x)$  evaluated at the random variables  $\{x_j \in \mathbb{R}_+ : j = 1, \dots, n\}$ , namely,

$$\sum_{j=1}^n f(x_j).$$

In our problem,  $f(x) > 0$ ,  $x \in \mathbb{R}_+$ .

Denote by  $M_f(t)$  the moment generating of the linear statistics. This is obtained by a Laplace transform of the probability density function,  $\mathbb{P}_f(Q)$ ,

$$M_f(t) = \int_{\mathbb{R}_+} \mathbb{P}_f(Q) e^{-tQ} dQ = \frac{\det(\mu_{j+k}(t))_{j,k=0}^{n-1}}{\det(\mu_{j+k}(0))_{j,k=0}^{n-1}},$$

where

$$\mu_j(t) := \int_{\mathbb{R}_+} x^j w(x) e^{-tf(x)} dx, \quad j \in \{0, 1, \dots\}.$$

Hence the moment generating function becomes the quotient of Hankel determinants.

For the problem at hand, the finite  $n$  version of which first appeared in [20], where  $f(x) := 1/x$ , and  $w(x)$  is the Laguerre weight supported on  $\mathbb{R}_+$ :

$$w(x) = x^\alpha e^{-x}, \quad \alpha > 0.$$

Such a linear statistics leads to the perturbed Laguerre weight,

$$w(x; t, \alpha) = w(x) e^{-\frac{t}{x}} = x^\alpha e^{-x-t/x}, \quad x \in \mathbb{R}_+, \quad t \geq 0, \quad \alpha > 0, \quad (1.1)$$

and the perturbed Hankel determinant:

$$D_n(t, \alpha) = \det \left( \int_{\mathbb{R}_+} x^{j+k} w(x; t, \alpha) dx \right)_{j,k=0}^{n-1}. \quad (1.2)$$

Such a perturbation is motivated in part by a finite temperature integrable quantum field theory [35]. The Hankel determinant given by (1.2) maybe interpreted as the generating function for the distribution function of the Wigner delay time in chaotic cavities and recently studied in [40] through large deviation techniques. Mathematically,  $f(x) = 1/x$ , introduces an infinitely strong zero on the weight function at 0, and has the effect of pushing the left end point of the equilibrium density, (or charge density, if we view the eigenvalues as charges), away from 0 at a slow speed. If  $a$  is left end point of the support of the density, then for  $t > 0$ , and large  $n$ ,  $a = O\left(\frac{t^{2/3}}{n^{1/3}}\right)$ . See the top equation of page 16.

For the Laguerre weight,  $x^\alpha e^{-x}$ ,  $\alpha > 0$ ,  $x \in \mathbb{R}_+$ ,  $a = O\left(\frac{\alpha^2}{n}\right)$ .

One finds, in applications, other forms of  $f(x)$ . For example, the characterization of Shannon capacity [25] in the study of the outage and error probability in wireless communication. In this situation, the Laguerre weight is multiplied by  $\exp(-\lambda \ln(1 + x/t))$ . Here  $\lambda$  is the parameter which “generates” the Shannon capacity and  $t(> 0)$  plays the role of the time variable in the ensuing Painleve V equation. See Basor and Chen [3], for a recent review of this and other related matters. A similar multiplicative factor on the Jacobi weight,  $x^\alpha(1-x)^\beta$ ,  $0 \leq x \leq 1$  for  $\alpha = \pm 1/2$ , and  $\beta = \pm 1/2$ , arose in enumeration problems related to the moduli space of super-symmetric QCD in the Veneziano limit [21]. The Coulomb Fluid interpretation of the eigenvalues has been adopted to compute statistical properties involving large deviations [36].

In this paper we shall be concerned with a double scaling scheme, where  $t \rightarrow 0^+$ , and  $n \rightarrow \infty$ , such that  $s := (2n + 1 + \alpha)t$  is finite. And ultimately provide a description of the (double-scaled) Hankel determinant.

The remainder of this paper is organized as follows. Section 2 recalls elementary facts about orthogonal polynomials, a description of the Dyson Coulomb Fluid, followed by

recalling certain results obtained in Chen and Its [20]. We show that that double-scaled and in some sense infinite dimensional Hankel determinant, has an integral representation in terms of a  $C$ -potential, which satisfies a second order non-linear ode in  $s$ , which is equivalent to a “lesser”  $P_{III}$ . The  $\sigma$  function of Jimbo-Miwa-Okamoto associated with this particular  $P_{III}$  is also found. In section 3, we compute formal power series for the small  $s$  and large  $s$  behavior of the  $C(s)$ . In section 4, an evaluation is made on the constant term of the monic polynomials orthogonal with respect to  $w(x; t, \alpha)$ , namely,  $P_n(0; t, \alpha)$ , for large  $n$  and  $s = 2nt$ . We obtained relationship between certain constants, and from which the value of the constant of the asymptotic expansion of the double-scaled Hankel determinant is conjectured. We conclude in section 5.

## 2 Double Scaling.

In this section we study the effect of double scaling by sending  $n \rightarrow \infty$  and  $t \rightarrow 0^+$  and such that  $s := (2n + 1 + \alpha)t$  is fixed.

To set the stage for later development, we recall for the Reader that the Hankel determinant can also be expressed as

$$D_n(t, \alpha) = \prod_{j=0}^{n-1} h_j(t), \quad (2.3)$$

where  $\{h_j(t) : j = 0, \dots, n-1\}$  are the squares of the  $L^2$  norm of the monic polynomials  $P_n(x)$  orthogonal with respect to  $w(x; t, \alpha)$ , namely,

$$\int_{\mathbb{R}_+} P_j(x) P_k(x) w(x; t, \alpha) dx = h_j(t) \delta_{jk}. \quad (2.4)$$

The monic polynomials satisfy the recurrence relations,

$$xP_n(x) = P_{n+1}(x) + \alpha_n(t)P_n(x) + \beta_n(t)P_{n-1}(x),$$

subject to the initial data,  $P_0(x) = 1$ , and  $\beta_0 P_{-1}(x) = 0$ . It is clear that the coefficients of the polynomials  $P_n(x)$  and the recurrence coefficients  $\alpha_n$ ,  $\beta_n$  all depend on  $t$ . We shall see later that  $\alpha_n(t)$ , the diagonal recurrence coefficient plays an important role.

Heine’s formula, gives the multiple integral representation of the orthogonal polynomials,

$$P_n(z; t, \alpha) = \frac{1}{D_n(t, \alpha)} \frac{1}{n!} \int_{\mathbb{R}_+^n} \prod_{m=1}^n (z - x_m) \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^n w(x_\ell; t, \alpha) dx_\ell,$$

see Szegő [39]. Here, the Hankel determinant reads,

$$D_n(t, \alpha) = \det \left( \int_{\mathbb{R}_+} x^{i+j} w(x; t, \alpha) dx \right)_{0 \leq i, j \leq n-1},$$

$$= \frac{1}{n!} \int_{\mathbb{R}_+^n} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^n w(x_\ell; t, \alpha) dx_\ell.$$

It is clear that,

$$(-1)^n P_n(0; t, \alpha) = \frac{D_n(t, \alpha + 1)}{D_n(t, \alpha)}, \quad (2.5)$$

and for the *monic* Laguerre polynomials, namely,  $P_n(z; 0, \alpha)$ , there is a closed form evaluation at  $z = 0$ , see ((5.1.7), [39]),

$$(-1)^n P_n(0; 0, \alpha) = \frac{\Gamma(n + 1 + \alpha)}{\Gamma(1 + \alpha)}, \quad \alpha > -1. \quad (2.6)$$

We give a brief description of Coulomb Fluid. The energy of a system of  $n$  logarithmically repelling particles on the line, confined by an external potential  $v$  reads

$$E(x_1, x_2, \dots, x_n) = -2 \sum_{1 \leq j < k \leq n} \ln |x_j - x_k| + \sum_{j=1}^n v(x_j).$$

For sufficiently large  $n$ , the particles may be approximated by a continuous fluid[31], with a density  $\sigma$ . In the Coulomb fluid approximation,  $\sigma(x)$ , assumed to be supported on  $[a, b]$  is obtained by minimizing the free-energy functional,  $F[\sigma]$ ,

$$\min_{\sigma > 0} F[\sigma] \quad \text{subject to} \quad \int_a^b \sigma(x) dx = n,$$

where

$$F[\sigma] := \int_a^b \sigma(x) v(x) dx - \int_a^b \int_a^b \sigma(x) \ln |x - y| \sigma(y) dx dy.$$

Upon minimization, the density  $\sigma(x)$  is found to satisfy the integral equation,

$$A = v(x) - 2 \int_a^b \ln |x - y| \sigma(y) dy, \quad x \in [a, b], \quad (2.7)$$

where  $A$ , the Lagrange multiplier imposes the constraint  $\int_a^b \sigma(x) dx = n$ . For more information, see [17] and references therein. Note that  $A$  is a constant independent of  $x$  for  $x \in [a, b]$ , but  $A$  and  $\sigma$  depend on  $t$  and  $n$ . The relations between Coulomb fluid and orthogonal polynomials, where the potential  $v$ , being convex, can be found, for example, in [17]. A description of the density for exponential weights and strong asymptotics for the orthogonal polynomials can be found in [30]. For further information on the equilibrium density see [28].

For the problem at hand, the density, supported on  $[a, b]$  reads,

$$\sigma(x) = \frac{\sqrt{(b-x)(x-a)}}{2\pi} \left[ \left( \frac{\alpha}{\sqrt{ab}} + \frac{t(a+b)}{2(ab)^{\frac{3}{2}}} \right) \frac{1}{x} + \frac{t}{x^2 \sqrt{ab}} \right], \quad a \leq x \leq b.$$

This is obtained by solving a singular integral equation, found by taking a derivative with respect to  $x$  on (2.7);

$$v'(x) - 2P \int_a^b \frac{\sigma(y)}{x-y} dy = 0, \quad x \in [a, b].$$

Here  $P$  denotes the Cauchy principal value, with the condition that the equilibrium density,  $\sigma$ , vanishes at the end points of the support.

See [17, 22], for further discussion. For this problem  $v(x) = -\ln w(x; t, \alpha)$ , and

$$v'(x) = -\frac{\alpha}{x} + 1 - \frac{t}{x^2}.$$

The end points of the interval  $[a, b]$  are determined by the normalization condition

$$\int_a^b \sigma(x) dx = n,$$

and a supplementary conditions, which can be found for example, in [17] and [22]. These are

$$\int_a^b \frac{xv'(x)}{\sqrt{(b-x)(x-a)}} dx = 2\pi n,$$

and

$$\int_a^b \frac{v'(x)}{\sqrt{(b-x)(x-a)}} dx = 0.$$

With the aid of the integrals in Appendix A, the end points of the support,  $a$  and  $b$ , satisfy algebraic equations,

$$2n + \alpha + \frac{t}{\sqrt{ab}} = \frac{a+b}{2}, \quad (2.8)$$

and

$$\frac{(a+b)t}{2(ab)^{\frac{3}{2}}} + \frac{\alpha}{\sqrt{ab}} = 1. \quad (2.9)$$

We state a lemma here, which gives further insight into a particular  $P_{III}(-4(2n+1+\alpha), -4\alpha, 4, -4)$ , that appeared in the finite  $n$  setting.

**Lemma 1.** *The geometric mean of  $a$  and  $b$ , namely,  $\tilde{X} := \sqrt{ab}$  satisfies the quartic equation,*

$$\tilde{X}^4 - \alpha \tilde{X}^3 - (2n + \alpha)t \tilde{X} - t^2 = 0. \quad (2.10)$$

*Proof.* The equation (2.10) is found by eliminating  $\frac{a+b}{2}$  from (2.8) and (2.9).  $\square$

In order to characterize the large  $n$  behavior of the Hankel determinant, we recall certain results in [20]. For convenience, we use  $t$  and  $y_n(t)$  instead of  $s$  and  $a_n(s)$  adopted in [20]. See (Theorem 1, [20]).

**Theorem 1.** *The diagonal recurrence coefficients,  $\alpha_n(t)$ , maybe expressed as,*

$$\alpha_n(t) = 2n + 1 + \alpha + y_n(t),$$

where the auxiliary quantity  $y_n(t)$ ,  $n = 0, 1, 2, \dots$  satisfies

$$y_n'' = \frac{(y_n')^2}{y_n} - \frac{y_n'}{t} + (2n + 1 + \alpha) \frac{y_n^2}{t^2} + \frac{y_n^3}{t^2} + \frac{\alpha}{t} - \frac{1}{y_n}, \quad (2.11)$$

with the initial conditions

$$y_n(0) = 0, \quad y_n'(0) = \frac{1}{\alpha}, \quad \alpha > 0. \quad (2.12)$$

If  $y_n(t) := -q(t)$ , then  $q(t)$  is a solution of  $P_{\text{III}}(-4(2n + 1 + \alpha), -4\alpha, 4, -4)$ , following the convention of [38].

Substituting

$$y_n(t) := \frac{t}{X_n(t)},$$

into (2.11) it is seen that  $X_n(t)$ , satisfies,

$$X_n'' = \frac{(X_n')^2}{X_n} - \frac{X_n'}{t} - \frac{\alpha(X_n)^2}{t^2} - \frac{2n + 1 + \alpha}{t} + \frac{X_n^3}{t^2} - \frac{1}{X_n}, \quad (2.13)$$

with the boundary condition,

$$X_n(0) = \alpha, \quad \alpha > 0,$$

which is recognized to be an equivalent  $P_{\text{III}}(-4\alpha, -4(2n + 1 + \alpha), 4, -4)$ .

If we disregard the derivatives in (2.13), then  $X_n$  satisfies a quartic equation,

$$X_n^4 - \alpha X_n^3 - (2n + 1 + \alpha)t X_n - t^2 = 0.$$

Note that the quartic obtained above becomes the quartic of (2.10), if we replace  $2n + 1$  by  $2n$ . Theorem 2 in [20] reveals an important relation between  $X_n$  and the logarithmic derivative of the Hankel determinant which we recall in the following Theorem.

**Theorem 2.**

$$\ln \frac{D_n(t, \alpha)}{D_n(0, \alpha)} = \int_0^t \left( \frac{\xi}{2} - \frac{1}{4} (X_n - \alpha)^2 - (n + \frac{\alpha}{2}) \frac{\xi}{X_n} - \frac{\xi^2}{4X_n^2} + \frac{\xi^2 (X'_n)^2}{4X_n^2} \right) \frac{d\xi}{\xi}, \quad (2.14)$$

where  $D_n(0, \alpha)$  has a closed form evaluation;

$$D_n(0, \alpha) = \frac{G(n+1)G(n+\alpha+1)}{G(\alpha+1)}.$$

Here  $G(z)$ ,  $z \in \mathbb{C} \cup \{\infty\}$  is the Barnes  $G$ -function, an entire function of order 2, and satisfies the functional relation  $G(z+1) = \Gamma(z) G(z)$ , where  $G(1) = 1$ .

The next Theorem recalls equations (3.23) and (3.24) in [20].

**Theorem 3.** *If*

$$H_n(t) := t \frac{d}{dt} \ln \frac{D_n(t, \alpha)}{D_n(0, \alpha)}, \quad (2.15)$$

then

$$(tH_n'')^2 = [n - (2n + \alpha)H_n']^2 - 4[n(n + \alpha) + tH_n' - H_n] H_n' (H_n' - 1), \quad (2.16)$$

subject to the initial condition,  $H_n(0) = 0$ .

The equation (2.16) is the Jimbo-Miwa-Okamoto  $\sigma$ -form of  $P_{\text{III}}$ , see [33]. From (2.14) and (2.15), we see that  $H_n(t)$  maybe expressed in terms of  $X_n(t)$  as

$$H_n = \frac{t}{2} - \frac{1}{4} (X_n - \alpha)^2 - (n + \frac{\alpha}{2}) \frac{t}{X_n} - \frac{t^2}{4X_n^2} + \frac{t^2 (X'_n)^2}{4X_n^2}. \quad (2.17)$$

Recall the double scaling process: Sending  $n \rightarrow \infty$  and  $t \rightarrow 0^+$  in such a way that  $s := (2n + 1 + \alpha)t$  is fixed.

In the next theorem, we find that it is convenient to introduce a “potential” defined by

$$C(s) = \lim_{n \rightarrow \infty} \frac{y_n \left( \frac{s}{2n+1+\alpha} \right)}{\frac{s}{2n+1+\alpha}} = \lim_{n \rightarrow \infty} \frac{1}{X_n \left( \frac{s}{2n+1+\alpha} \right)}. \quad (2.18)$$

**Theorem 4.** *Let*

$$s := (2n + 1 + \alpha)t,$$

where  $t \rightarrow 0^+$  and  $2n + 1 + \alpha \rightarrow \infty$ , and such that  $s$  is finite. Let

$$\Delta(s, \alpha) := \lim_{n \rightarrow \infty} \frac{D_n \left( \frac{s}{2n+1+\alpha}, \alpha \right)}{D_n(0, \alpha)}, \quad \text{where } \Delta(0, \alpha) = 1. \quad (2.19)$$



The  $C$  potential, satisfies

$$C''(s) = \frac{(C'(s))^2}{C} - \frac{C'(s)}{s} + \frac{(C(s))^2}{s} + \frac{\alpha}{s^2} - \frac{1}{s^2 C(s)}, \quad (2.20)$$

with the initial condition  $C(0) = 1/\alpha$ . If

$$\mathcal{H}(s) := \lim_{n \rightarrow \infty} H_n(s/(2n+1+\alpha)) = s \frac{d}{ds} \ln \Delta(s, \alpha),$$

then  $\mathcal{H}(s)$  satisfies

$$(s\mathcal{H}'')^2 + 4(\mathcal{H}')^2 (s\mathcal{H}' - \mathcal{H}) - \left( \alpha \mathcal{H}' + \frac{1}{2} \right)^2 = 0, \quad (2.21)$$

subject to the initial condition,  $\mathcal{H}(0) = 0$ . Moreover,

$$\mathcal{H}(s) = \frac{1}{4} \left( \frac{sC'(s)}{C(s)} \right)^2 - \frac{sC(s)}{2} - \frac{1}{4} \left( \frac{1}{C(s)} - \alpha \right)^2. \quad (2.22)$$

*Proof.* By a straight forward, formal, if tedious computations, we see that (2.13) becomes (2.20), (2.16) becomes (2.21), and (2.17) becomes (2.22), after double scaling.  $\square$

**Remark 1.** From (2.14) and (2.19), we have,

$$\ln \Delta(s, \alpha) = \int_0^s \left\{ \frac{1}{4} \left( \frac{\xi}{C(\xi)} \frac{dC(\xi)}{d\xi} \right)^2 - \frac{\xi C(\xi)}{2} - \frac{1}{4} \left( \frac{1}{C(\xi)} - \alpha \right)^2 \right\} \frac{d\xi}{\xi}. \quad (2.23)$$

**Remark 2.** By a change of variables  $Y(x) = \frac{x}{2}C(\frac{x^2}{8})$ ,  $x \in (0, \infty)$ , the equation (2.20), becomes a “lesser”  $P_{III}$ ;

$$Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{x} + \frac{Y^2}{x} - \frac{1}{Y} + \frac{2\alpha}{x}. \quad (2.24)$$

From [38], we see that (2.24) is  $P_{III}(1, 2\alpha, 0, -1)$ ; the  $P_{III}$  with a fewer number of parameters, mentioned in the abstract.

**Remark 3.** Substituting

$$\mathbb{H}(s) =: \mathcal{H}(2s) - \frac{\alpha^2}{4},$$

into (2.21), we find,

$$(s\mathbb{H}''(s))^2 + 4\mathbb{H}'^2(s) (s\mathbb{H}'(s) - \mathbb{H}(s)) - 2\alpha \mathbb{H}'(s) - 1 = 0,$$

which is the  $P_{III}$  obtained by Ohyama-Kawamuko-Sakai-Okamoto, see ((18), in [38]), but with  $\alpha_1$  replaced by  $\alpha$ .

## 2.1 Coulomb Fluid continued.

Recall equation (2.10) derived by the Coulomb Fluid method,

$$\tilde{X}^4 - \alpha \tilde{X}^3 - (2n + \alpha)t \tilde{X} - t^2 = 0.$$

Substituting  $t = s/(2n + \alpha)$ , into quartic, following by  $n \rightarrow \infty$ , we find,

$$\tilde{X}^3 - \alpha \tilde{X}^2 - s = 0,$$

a cubic equation in  $\tilde{X}$ .

Let

$$\tilde{C} = \frac{1}{\tilde{X}},$$

this cubic becomes,

$$\frac{\tilde{C}^2}{s} + \frac{\alpha}{s^2} - \frac{1}{s^2 \tilde{C}} = 0.$$

Note that this is the “algebraic part” of (2.20).

Retaining only the real solution of the cubic equation in  $\tilde{C}$ , we find,

$$\tilde{C}(s) = -2^{\frac{1}{3}} \alpha \left[ 27s^2 + (729s^4 + 108\alpha^3 s^3)^{\frac{1}{2}} \right]^{-\frac{1}{3}} + 18^{-\frac{1}{3}} s^{-1} \left[ 9s^2 + (81s^4 + 12s^3 \alpha^3)^{\frac{1}{2}} \right]^{\frac{1}{3}}.$$

A Taylor series expansion of  $\tilde{C}(s)$  about  $s = 0$  gives, for  $\alpha > 0$ ,

$$\tilde{C}(s) = \frac{1}{\alpha} - \frac{1}{\alpha^4} s + \frac{3}{\alpha^7} s^2 - \frac{12}{\alpha^{10}} s^3 + \frac{55}{\alpha^{13}} s^4 + O(s^5). \quad (2.25)$$

For large and positive  $s$ , one finds

$$\tilde{C}(s) = s^{-\frac{1}{3}} - \frac{\alpha}{3} s^{-\frac{2}{3}} + \frac{\alpha^3}{81} s^{-\frac{4}{3}} + \frac{\alpha^4}{243} s^{-\frac{5}{3}} - \frac{4\alpha^6}{6561} s^{-\frac{7}{3}} + O(s^{-\frac{8}{3}}). \quad (2.26)$$

In the next section, we derive the small  $s$  and large  $s$  expansion of the  $C$  potential, assuming appropriate forms of the expansions.

## 3 Small $s$ and large $s$ behavior of the $C$ potential.

We consider real-valued solutions of the Painlevé equations throughout this paper.

We study the solution of the  $C$  potential for  $s \rightarrow 0^+$ , by substituting

$$C(s) = \sum_{j=0}^{\infty} a_j s^j,$$

into (2.20). We find  $a_0 = 1/\alpha$ , and  $a_1 = -1/(\alpha^2(\alpha^2 - 1))$ . For  $n \geq 2$ , the coefficients satisfy a recurrence relation,

$$\sum_{j=0}^n j(j-1)a_j a_{n-j} - \sum_{j=0}^n j(n-j)a_j a_{n-j} + \sum_{j=0}^n j a_j a_{n-j} - \sum_{j=0}^{n-1} \sum_{k=0}^{n-1-j} a_j a_k a_{n-1-j-k} - \alpha a_n = 0.$$

A straightforward computation, gives,

$$\begin{aligned} C(s) = & \frac{1}{\alpha} - \frac{1}{\alpha^2(\alpha^2 - 1)}s + \frac{3}{\alpha^3(\alpha^2 - 1)(\alpha^2 - 4)}s^2 - \frac{6(2\alpha^2 - 3)}{\alpha^4(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)}s^3 \\ & + \frac{5(-36 + 11\alpha^2)}{\alpha^5(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)(\alpha^2 - 16)}s^4 \\ & + \frac{3(3600 - 4219\alpha^2 + 1115\alpha^4 - 91\alpha^6)}{\alpha^6(\alpha^2 - 1)^3(\alpha^2 - 4)^2(\alpha^2 - 9)(\alpha^2 - 16)(\alpha^2 - 25)}s^5 + O(s^6), \end{aligned} \quad (3.27)$$

where  $\alpha \notin \mathbb{Z}$ .

For large and positive  $s$ , we substitute the asymptotic expansion

$$C(s) = \sum_{k=1}^{\infty} b_k s^{-\frac{k}{3}},$$

into (2.20), and find  $b_j : j = 1, 2, \dots, n$  satisfies

$$\frac{1}{9} \sum_{j=1}^n j(2j+3-n)b_j b_{n-j} - \frac{1}{3} \sum_{j=1}^n j b_j b_{n-j} - \sum_{j=1}^{n+3} \sum_{k=1}^{n+3-j} b_j b_k b_{n+3-j-k} - \alpha b_n = 0, \quad n \geq 2,$$

$$\text{with } 1 - b_1^3 = 0, \text{ and } \sum_{j=1}^4 \sum_{k=1}^{4-j} b_j b_k b_{4-j-k} + \alpha b_1 = 0.$$

Continuing,

$$\begin{aligned} C(s) = & s^{-\frac{1}{3}} - \frac{\alpha}{3}s^{-\frac{2}{3}} + \frac{\alpha(\alpha^2 - 1)}{81}s^{-\frac{4}{3}} + \frac{\alpha^2(\alpha^2 - 1)}{243}s^{-\frac{5}{3}} + \frac{\alpha(\alpha^2 - 1)}{243}s^{-2} \\ & - \frac{2\alpha^2(\alpha^2 - 1)(2\alpha^2 - 11)}{6561}s^{-\frac{7}{3}} - \frac{5\alpha(\alpha^2 - 1)(\alpha^4 - \alpha^2 - 15)}{19683}s^{-\frac{8}{3}} + O(s^{-3}). \end{aligned} \quad (3.28)$$

**Remark 4.** Note that for  $\alpha = 0, \pm 1$ , the asymptotic expansion terminates. The three relevant algebraic solutions for the  $C$  potential are,

$$\begin{aligned} C(s) &= s^{-\frac{1}{3}}, \quad \alpha = 0 \\ C(s) &= s^{-1/3} \mp \frac{1}{3}s^{-\frac{2}{3}}, \quad \alpha = \pm 1. \end{aligned}$$

**Remark 5.** The series expansion for the  $C$  potential, given in (3.27), for sufficiently large  $|\alpha|$  is the same as the corresponding series for  $\tilde{C}$  derived by Coulomb Fluid method. This phenomenon also holds for the large  $s$  asymptotic, seen by comparing (3.28) with (2.26).

### 3.1 The behavior of $\mathcal{H}(s)$ for small $s$ and large $s$ .

Recall the  $\sigma$ -form of our Painlevé equation,

$$(s\mathcal{H}'')^2 + 4(\mathcal{H}')^2 (s\mathcal{H}' - \mathcal{H}) - \left(\alpha\mathcal{H}' + \frac{1}{2}\right)^2 = 0,$$

with the initial condition,  $\mathcal{H}(0) = 0$ .

For small  $s$ , we assume a power series expansion in  $s$ ,

$$\mathcal{H}(s) = \sum_{j=0}^{\infty} d_j s^j.$$

Substituting this into the equation satisfied by  $\mathcal{H}(s)$ , followed by some computation, gives

$$\begin{aligned} \mathcal{H}(s) = & -\frac{1}{2\alpha}s + \frac{1}{4\alpha^2(\alpha^2-1)}s^2 - \frac{1}{2\alpha^3(\alpha^2-1)(\alpha^2-4)}s^3 + \frac{3(2\alpha^2-3)}{4\alpha^4(\alpha^2-1)^2(\alpha^2-4)(\alpha^2-9)}s^4 \\ & + \frac{-36+11\alpha^2}{2\alpha^5(\alpha^2-1)^2(\alpha^2-4)(\alpha^2-9)(\alpha^2-16)}s^5 \\ & + \frac{-3600+4219\alpha^2-1115\alpha^4+91\alpha^6}{4\alpha^6(\alpha^2-1)^3(\alpha^2-4)^2(\alpha^2-9)(\alpha^2-16)(\alpha^2-25)}s^6 + O(s^7), \end{aligned} \quad (3.29)$$

where  $\alpha \notin \mathbb{Z}$ .

For positive and large  $s$ , we substitute the expansion

$$\mathcal{H}(s) = s^{\frac{2}{3}} \left( \sum_{j=0}^{\infty} \eta_j s^{-\frac{j}{3}} \right),$$

into the  $\sigma$ -form equation. After some computations, the expansions of  $\mathcal{H}(s)$ , for large and positive  $s$  read,

$$\begin{aligned} \mathcal{H}(s) = & -\frac{3}{4}s^{\frac{2}{3}} + \frac{\alpha}{2}s^{\frac{1}{3}} + \frac{1-6\alpha^2}{36} + \frac{\alpha(\alpha^2-1)}{54}s^{-\frac{1}{3}} + \frac{\alpha^2(\alpha^2-1)}{324}s^{-\frac{2}{3}} + \frac{\alpha(\alpha^2-1)}{486}s^{-1} \\ & - \frac{\alpha^2(\alpha^2-1)(2\alpha^2-11)}{8748}s^{-\frac{4}{3}} - \frac{\alpha(\alpha^6-2\alpha^4-14\alpha^2+15)}{13122}s^{-\frac{5}{3}} \\ & - \frac{8\alpha^6-41\alpha^4+33\alpha^2}{26244}s^{-2} + O(s^{-\frac{7}{3}}). \end{aligned} \quad (3.30)$$

The next Theorem gives the small  $s$  and large  $s$  expansions of  $\Delta(s, \alpha)$ .

**Theorem 5.** *In the double scaling  $n \rightarrow \infty$ ,  $t \rightarrow 0$ , such that  $s := (2n+1+\alpha)t$  and  $s \in (0, \infty)$ , the expressions of  $\Delta(s, \alpha)$ , for small  $s$  and large  $s$  are as follows:*

For small  $s$ ,

$$\begin{aligned} \Delta(s, \alpha) = \exp & \left[ -\frac{1}{2\alpha}s + \frac{1}{8\alpha^2(\alpha^2-1)}s^2 - \frac{1}{6\alpha^3(\alpha^2-1)(\alpha^2-4)}s^3 \right. \\ & + \frac{3(2\alpha^2-3)}{16\alpha^4(\alpha^2-1)^2(\alpha^2-4)(\alpha^2-9)}s^4 + \frac{36-11\alpha^2}{10\alpha^5(\alpha^2-1)^2(\alpha^2-4)(\alpha^2-9)(\alpha^2-16)}s^5 \\ & \left. + \frac{91\alpha^6-1115\alpha^4+4219\alpha^2-3600}{24\alpha^6(\alpha^2-4)^2(\alpha^2-1)^3(\alpha^2-9)(\alpha^2-16)(\alpha^2-25)}s^6 + O(s^7) \right], \end{aligned} \quad (3.31)$$

where  $\alpha \notin \mathbb{Z}$ .

For large  $s$ ,

$$\begin{aligned} \Delta(s, \alpha) = \exp & \left[ c_1 - \frac{9}{8}s^{\frac{2}{3}} + \frac{3\alpha}{2}s^{\frac{1}{3}} + \frac{1-6\alpha^2}{36}\ln s + \frac{\alpha(1-\alpha^2)}{18}s^{-\frac{1}{3}} + \frac{\alpha^2(1-\alpha^2)}{216}s^{-\frac{2}{3}} \right. \\ & + \frac{\alpha(1-\alpha^2)}{486}s^{-1} + \frac{\alpha^2(2\alpha^4-13\alpha^2+11)}{11664}s^{-\frac{4}{3}} + \frac{\alpha(\alpha^6-2\alpha^4-14\alpha^2+15)}{21870}s^{-\frac{5}{3}} \\ & \left. + O(s^{-3}) \right], \end{aligned} \quad (3.32)$$

where  $c_1 = c_1(\alpha)$  is a constant independent of  $s$ .

*Proof.* Recall (2.23),

$$\ln \Delta(s, \alpha) = \int_0^s \left\{ \frac{1}{4} \left( \frac{\xi}{C(\xi)} \frac{dC(\xi)}{d\xi} \right)^2 - \frac{\xi C(\xi)}{2} - \frac{1}{4} \left( \frac{1}{C(\xi)} - \alpha \right)^2 \right\} \frac{d\xi}{\xi}.$$

Substituting (3.27) and (3.28) into the above formula, then the expansions of  $\Delta(s, \alpha)$  for small  $s$ , (3.31) and for large  $s$ , (3.32) follow immediately.

The results obtained coincide with the integration of

$$\mathcal{H}(s) = s \frac{d}{ds} \ln \Delta(s, \alpha), \quad \mathcal{H}(0) = 0,$$

with the expansions of  $\mathcal{H}(s)$  for small  $s$ , (3.29) and for large  $s$ , (3.30).  $\square$

At the end of this section we compute the large  $n$  behavior of  $P_n(0; t, \alpha)$ , namely the evaluation of the orthogonal polynomials at the origin, from the fact that,

$$(-1)^n P_n(0; t, \alpha) = \frac{D_n(t, \alpha+1)}{D_n(t, \alpha)}.$$

Note the exact evaluation of  $(-1)^n P_n(0; 0, \alpha)$  in (2.6).

**Corollary 1.** *Under double scaling,*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(-1)^n P_n \left( 0; \frac{s}{2n+\alpha+1}, \alpha \right)}{(-1)^n P_n (0; 0, \alpha)} &= \frac{\Delta(s, \alpha+1)}{\Delta(s, \alpha)} = \exp \left( c_2 + \frac{3}{2} s^{\frac{1}{3}} - \frac{1+2\alpha}{6} \ln s - \frac{\alpha(\alpha+1)}{6} s^{-\frac{1}{3}} \right. \\ &\quad - \frac{\alpha(\alpha+1)(2\alpha+1)}{108} s^{-\frac{2}{3}} - \frac{\alpha(\alpha+1)}{162} s^{-1} \\ &\quad \left. + \frac{\alpha(\alpha+1)(2\alpha+1)(\alpha^2+\alpha-3)}{1944} s^{-\frac{4}{3}} + O(s^{-\frac{5}{3}}) \right), \end{aligned} \quad (3.33)$$

where  $c_2 = c_2(\alpha)$  is a constant independent of  $s$ .

*Proof.* From the fact that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(-1)^n P_n \left( 0; \frac{s}{2n+\alpha+1}, \alpha \right)}{(-1)^n P_n (0; 0, \alpha)} &= \lim_{n \rightarrow \infty} \frac{D_n(s/(2n+\alpha+1), \alpha+1)}{D_n(0, \alpha+1)} \frac{D_n(0, \alpha)}{D_n(s/(2n+\alpha+1), \alpha)} \\ &= \frac{\Delta(s, \alpha+1)}{\Delta(s, \alpha)}. \end{aligned}$$

Together with the expression of  $\Delta(s, \alpha+1)$  and  $\Delta(s, \alpha)$  given by (3.32); the equation (3.33) is obtained. Furthermore, there is a relation between  $c_1(\alpha)$  and  $c_2(\alpha)$ , namely,

$$c_2(\alpha) = c_1(\alpha+1) - c_1(\alpha). \quad (3.34)$$

□

In the next section a computation produces the constant  $c_2(\alpha)$ .

## 4 The asymptotic of $P_n(0; t, \alpha)$ .

In this section, we evaluate  $P_n(0; t, \alpha)$ , for large  $n$ , and give a derivation of  $c_2(\alpha)$ . First we state a result regarding the large  $n$  behaviour of the orthogonal polynomials  $P_n(z; t, \alpha)$ , for  $z \notin [a, b]$ . In our problem, the potential,  $v(x) = -\alpha \ln x + x + \frac{t}{x}$ ,  $x \geq 0, t > 0$  and  $\alpha > 0$ , satisfy the convexity condition[22].

The formulas below are valid for  $z \notin [a, b]$ , and large  $n$ .

For large  $n$ ,  $P_n(z)$  can be computed as

$$P_n(z) \sim \exp[-S_1(z) - S_2(z)], \quad \text{where } z \notin [a, b], \quad (4.35)$$

and  $S_1(z)$  and  $S_2(z)$  are given by ((4.6) and (4.7), [22]).

These are

$$S_1(z) = \frac{1}{4} \ln \left[ \frac{16(z-a)(z-b)}{(b-a)^2} \left( \frac{\sqrt{z-a} - \sqrt{z-b}}{\sqrt{z-a} + \sqrt{z-b}} \right)^2 \right], \quad z \notin [a, b],$$

and

$$S_2(z) = -n \ln \left( \frac{\sqrt{z-a} + \sqrt{z-b}}{2} \right)^2 + \frac{1}{2\pi} \int_a^b \frac{v(x)}{\sqrt{(b-x)(x-a)}} \left[ \frac{\sqrt{(z-a)(z-b)}}{x-z} + 1 \right] dx, \quad z \notin [a, b]. \quad (4.36)$$

We mention here an equivalent representation of  $S_1(z)$ ,  $z \notin [a, b]$ ,

$$\exp(-S_1(z)) = \frac{1}{2} \left[ \left( \frac{z-b}{z-a} \right)^{\frac{1}{4}} + \left( \frac{z-a}{z-b} \right)^{\frac{1}{4}} \right]. \quad (4.37)$$

The next theorem gives an evaluation of  $P_n(0; t, \alpha)$  for large  $n$ .

**Theorem 6.** *If  $v(x) = -\alpha \ln x + x + \frac{t}{x}$ ,  $x \geq 0$ ,  $t > 0$  and  $\alpha > 0$ , the evaluations at  $z = 0$  of  $S_1(z; t, \alpha)$ ,  $S_2(z; t, \alpha)$ , and  $P_n(z; t, \alpha)$  are given by*

$$\exp[-S_1(0; t, \alpha)] \sim (2t)^{-\frac{1}{6}} n^{\frac{1}{3}}, \quad (4.38)$$

$$(-1)^n \exp[-S_2(0; t, \alpha)] \sim n^n (2t)^{-\frac{\alpha}{3}} \exp \left( -n + 3 \cdot 2^{-\frac{2}{3}} n^{\frac{1}{3}} t^{\frac{1}{3}} + \frac{2\alpha}{3} \ln n \right), \quad (4.39)$$

and

$$\begin{aligned} (-1)^n P_n(0; t, \alpha) &\sim (-1)^n \exp[-S_1(0; t, \alpha) - S_2(0; t, \alpha)] \\ &\sim n^n (2t)^{-\frac{1}{6} - \frac{\alpha}{3}} \exp \left( -n + 3 \cdot 2^{-\frac{2}{3}} n^{\frac{1}{3}} t^{\frac{1}{3}} + \frac{1}{3} (1 + 2\alpha) \ln n \right), \end{aligned} \quad (4.40)$$

furthermore, the asymptotic estimates are uniform with respect to  $t \in (0, a_0]$ ,  $0 < a_0 < \infty$ ,  $\alpha > 0$ ,  $n \rightarrow \infty$  and  $nt \rightarrow \infty$ .

*Proof.* Recall the quartic equation satisfied by  $\tilde{X}$ ,

$$\tilde{X}^4 - \alpha \tilde{X}^3 - (2n + \alpha)t \tilde{X} - t^2 = 0,$$

where  $\tilde{X} = \sqrt{ab}$ .

Let  $\tilde{n} = 2n + \alpha$ . The relevant solution of the quartic, reads, for large  $\tilde{n}$ ,

$$\frac{1}{\tilde{X}} = (\tilde{n}t)^{-\frac{1}{3}} - \frac{\alpha}{3}(\tilde{n}t)^{-\frac{2}{3}} + \frac{\alpha^3}{81}(\tilde{n}t)^{-\frac{4}{3}} + \left( \frac{\alpha^4}{243} - \frac{t^2}{3} \right) (\tilde{n}t)^{-\frac{5}{3}} + O((\tilde{n}t)^{-2}),$$

where  $t \in (0, a_0]$ ,  $0 < a_0 < \infty$ ,  $\alpha > 0$ ,  $n \rightarrow \infty$ .

From (2.8) and (2.9), we see that

$$a = \tilde{n} + \frac{t}{\tilde{X}} - \sqrt{\left(\tilde{n} + \frac{t}{\tilde{X}}\right)^2 - \tilde{X}^2} = \frac{t^{\frac{2}{3}}}{2\tilde{n}^{\frac{1}{3}}} + \frac{\alpha t^{\frac{1}{3}}}{3\tilde{n}^{\frac{2}{3}}} + \frac{\alpha^2}{6\tilde{n}} + \frac{5\alpha^3}{81t^{\frac{1}{3}}\tilde{n}^{\frac{4}{3}}} + O(\tilde{n}^{-\frac{5}{3}}),$$

and

$$b = \tilde{n} + \frac{t}{\tilde{X}} + \sqrt{\left(\tilde{n} + \frac{t}{\tilde{X}}\right)^2 - \tilde{X}^2} = 2\tilde{n} + \frac{3t^{\frac{2}{3}}}{2\tilde{n}^{\frac{1}{3}}} - \frac{\alpha t^{\frac{1}{3}}}{\tilde{n}^{\frac{2}{3}}} - \frac{\alpha^2}{6\tilde{n}} - \frac{\alpha^3}{27t^{\frac{1}{3}}\tilde{n}^{\frac{4}{3}}} + O(\tilde{n}^{-\frac{5}{3}}),$$

where  $t \in (0, a_0]$ ,  $0 < a_0 < \infty$ ,  $\alpha > 0$ ,  $n \rightarrow \infty$ .

Hence

$$ab = (\tilde{n}t)^{\frac{2}{3}} + \frac{2}{3}\alpha(\tilde{n}t)^{\frac{1}{3}} + O(1), \quad \text{which implies,} \quad \sqrt{ab} = (\tilde{n}t)^{\frac{1}{3}} + \frac{\alpha}{3} + O(n^{-\frac{1}{3}}).$$

From the expression for  $\exp(-S_1(z))$ , see (4.37), we find,

$$\exp[-S_1(0; t, \alpha)] \sim \frac{1}{2} \left(\frac{b}{a}\right)^{\frac{1}{4}} \sim 2^{-\frac{1}{6}} n^{\frac{1}{3}} t^{-\frac{1}{6}}. \quad (4.41)$$

We now evaluate  $S_2(0; t, \alpha)$  by setting  $z = 0$  in (4.36). With the aid of the integral identities in Appendix A, followed by some computations, we find,

$$\begin{aligned} \exp[-S_2(0; t, \alpha)] &= (-1)^n \left(n + \frac{\alpha}{2} + \frac{t}{2\sqrt{ab}} + \frac{\sqrt{ab}}{2}\right)^n \left(\frac{n}{\sqrt{ab}} + \frac{\alpha}{2\sqrt{ab}} + \frac{t}{2ab} + \frac{1}{2}\right)^\alpha \\ &\quad \times \exp \left[ -n - \frac{\alpha}{2} - \frac{t}{\sqrt{ab}} + \frac{\sqrt{ab}}{2} + \left(n + \frac{\alpha}{2}\right) \frac{t}{ab} + \frac{t^2}{2(ab)^{\frac{3}{2}}} \right]. \end{aligned}$$

Now since  $\sqrt{ab} \sim (\tilde{n}t)^{\frac{1}{3}} + \frac{\alpha}{3}$ , the above becomes (4.39). With the expressions for  $\exp(-S_1(0; t, \alpha))$  and  $\exp(-S_2(0; t, \alpha))$ , the asymptotic estimations for  $P_n(0; t, \alpha)$ , namely, (4.40) follows immediately.  $\square$

**Remark 6.** For convenience, we rewrite the asymptotic estimation of  $(-1)^n P_n(0; t, \alpha)$ , (4.40), as

$$\begin{aligned} (-1)^n P_n(0; t, \alpha) &\sim (-1)^n \exp[-S_1(0; t, \alpha) - S_2(0; t, \alpha)] \\ &\sim n^n (2t)^{-\frac{1}{6} - \frac{\alpha}{3}} \exp \left( -n + 3 \cdot 2^{-\frac{2}{3}} n^{\frac{1}{3}} t^{\frac{1}{3}} + \frac{1}{3} (1 + 2\alpha) \ln n \right) \\ &= n^{n+\alpha+\frac{1}{2}} e^{-n} \exp \left( \frac{3}{2} (2nt)^{\frac{1}{3}} - \frac{1+2\alpha}{6} \ln(2nt) \right) \\ &\sim \Gamma(n+1+\alpha) \exp \left( -\frac{1}{2} \ln(2\pi) + \frac{3}{2} (2nt)^{\frac{1}{3}} - \frac{1+2\alpha}{6} \ln(2nt) \right) \\ &= \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)} \exp \left( \ln \left( \frac{\Gamma(1+\alpha)}{\sqrt{2\pi}} \right) + \frac{3}{2} (2nt)^{\frac{1}{3}} - \frac{1+2\alpha}{6} \ln(2nt) \right), \end{aligned} \quad (4.42)$$



and hence (noting the exact evaluation—(2.6)),

$$\frac{P_n(0; t, \alpha)}{P_n(0; 0, \alpha)} \sim \exp \left( c_2(\alpha) + \frac{3}{2} (2nt)^{1/3} - \frac{1+2\alpha}{6} \ln(2nt) \right),$$

from which  $c_2(\alpha)$  is identified to be

$$\ln \left( \frac{\Gamma(1+\alpha)}{\sqrt{2\pi}} \right).$$

In the pan-ultimate step, leading to (4.42),

$$\sqrt{2\pi} e^{(n+\alpha+1/2) \ln n - n}$$

is replaced by

$$\Gamma(n+1+\alpha).$$

The singular perturbation, obtained through the multiplication of  $e^{-t/x}$  on Laguerre weight,  $x^\alpha e^{-x}$  causes a large distortion, in such a way that,

$$\frac{P_n(0; t, \alpha)}{P_n(0; 0, \alpha)} \sim \exp \left( \frac{3}{2} (2nt)^{1/3} - \frac{1+2\alpha}{6} \ln(2nt) + c_2(\alpha) \right).$$

Identifying  $2nt$  by  $s$ , then (4.42) becomes (3.33).

Hence, the equation (3.34) reads,

$$c_1(\alpha+1) - c_1(\alpha) = c_2(\alpha) = \ln \left( \frac{\Gamma(1+\alpha)}{\sqrt{2\pi}} \right), \quad \alpha > 0.$$

Noting the properties of the Barnes- $G$  function, gives,

$$c_1(\alpha) = \ln \frac{G(\alpha+1)}{(2\pi)^{\alpha/2}}.$$

However, a rigorous determination of  $c_1(\alpha)$  remains open.

## 5 Conclusion

The (finite  $n$ ) Hankel determinant generated by a singularly deformed Laguerre weight,  $x^\alpha \exp(-x - t/x)$ ,  $\alpha > -1$ ,  $t \geq 0$ ,  $x \geq 0$ , was shown to be intimately related to a particular (finite  $n$ )  $P_{III}$ . In the double scaling scheme considered in this paper, the *infinite Hankel determinant*, has an integral representation in terms of a  $C$  potential, which satisfies a second order non-linear ode. Up to minor changes of variable, this is equivalent to a  $P_{III}$  with smaller number of parameters. Asymptotic expansion of the *scaled* Hankel determinant are found, through the relevant Painlevé equations.

## Appendix A: Some Integration Identities.

The integrals listed below, valid for  $0 < a < b$ , can be found in the [32], and in [16] and [25]:

$$\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}} = \pi. \quad (A1)$$

$$\int_a^b \frac{x dx}{\sqrt{(b-x)(x-a)}} = \frac{(a+b)\pi}{2}. \quad (A2)$$

$$\int_a^b \frac{dx}{x\sqrt{(b-x)(x-a)}} = \frac{\pi}{\sqrt{ab}}. \quad (A3)$$

$$\int_a^b \frac{dx}{x^2\sqrt{(b-x)(x-a)}} = \frac{(a+b)\pi}{2(ab)^{\frac{3}{2}}}. \quad (A4)$$

$$\int_a^b \frac{\ln x}{\sqrt{(b-x)(x-a)}} dx = 2\pi \ln \left( \frac{\sqrt{a} + \sqrt{b}}{2} \right). \quad (A5)$$

$$\int_a^b \frac{\ln x}{x\sqrt{(b-x)(x-a)}} dx = \frac{2\pi}{\sqrt{ab}} \ln \frac{2\sqrt{ab}}{\sqrt{a} + \sqrt{b}}. \quad (A6)$$

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## References

- [1] M. Adler, P. Van Moerbeke, *Hermitian, symmetric and symplectic random ensembles: PDEs for the distribution of the spectrum*, Ann. of Math. 153 (2001), 149-189.
- [2] W. C. Bauldry, *Estimate of asymmetric Freud polynomials on the real line*, J. Approx. Theory 63 (1990), 225–237.
- [3] E. Basor, Y. Chen, *Perturbed Laguerre unitary ensembles, Hankel determinants and information theory*, Math. Meth. in the Appl. Scie. DOI:10.1002/mma, (2014).
- [4] E. Basor, Y. Chen, *Perturbed Hankel determinants*, J. Phys. A: Math. Gen. 38 (2005), 10101-10106.
- [5] E. Basor, Y. Chen, *Painlevé V and the distribution function of a discontinuous linear statistic in the Laguerre unitary ensembles*, J. Phys. A: Math. Theo. 42 (2009), 035203.

- [6] E. Basor, Y. Chen and H. Widom, *Hankel determinants as Fredholm determinants: Random Matrix Models and Their Applications*, MSRI publications, Camb. Univ. Press, Cambridge 40 (2001), 21–29.
- [7] E. Basor, Y. Chen and H. Widom, *Determinants of Hankel matrices*, J. Func. Anal. 179 (2001), 214–234.
- [8] S. Belmehdi, A. Ronveaux, *Laguerre-Freud’s equations for the recurrence coefficients of semi-classical orthogonal polynomials*, J. Approx. Thoery 76 (1994), 351–368.
- [9] M. Bertola, *Moment determinants as isomonodromic tau functions*, Nonlinearity 22 (2009), 29–50.
- [10] M. Bertola, B. Eynard, J. Harnad, *Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions*, Commun. Math. Phys. 263 (2006), 401–437.
- [11] S. Bonan, D. S. Lubinsky and P. Nevai, *Orthogonal polynomials and their derivative II*, SIAM J. Math. Anal. 18 (1987), 1163–1176.
- [12] S. Bonan, P. Nevai, *Orthogonal polynomials and their derivatives I*, J. Appro. Thoery 40 (1984), 134–147.
- [13] S. S. Bonan, D. S. Clark, *Estimates of the orthogonal polynomials with weight  $\exp(-x^m)$ ,  $m$  an even positive integer*, J. Approx. Theory 46 (1986), 408–410.
- [14] S. S. Bonan, D. S. Clark, *Estimates of the Hermite and Freud polynomials*, J. Approx. Theory 63 (1990), 210–224.
- [15] Y. Chen, M. V. Feigin, *Painlevé IV and degenerate Gaussian unitary ensembles*, J. Phys. A: Math. Gen. 39 (2006), 12381–12393.
- [16] Y. Chen, N. S. Haq, M. R. Mckay, *Random matrix models, double-time Painlevé equations, and wireless relaying*, J. of Math. Phys. vol.54, no.6 (2013), 063506.
- [17] Y. Chen, M. E. H. Ismail, *Thermodynamic relations of the Hermitian matrix ensembles*, J. Phys. A.: Math. Gen. vol.30, no.19 (1997), 6633–6654.
- [18] Y. Chen, M. E. H. Ismail, *Ladder operator and differential equations for orthogonal polynomials*, J. Phys. A: Math. Gen. 30 (1997), 7817–7829.
- [19] Y. Chen, M. Ismail, *Jacobi polynomials form compatibility conditions*, Proc. Amer. Math. Soc. 133 (2005), 465–472.
- [20] Y. Chen, A. Its, *Painlevé III and a singular linear statistics in Hermitian random matrix ensembles, I*, J. Appro. theory 162 (2010), 270–297.
- [21] Y. Chen, N. Jokela, M. Järvinen, N. Mekareeya, *Moduli space of supersymmetric QCD in the Veneziano limit*, J. High Ener. Phys. 9 (2013), 1–38.

- [22] Y. Chen, N. Lawrence, *On the linear statistics of Hermitian random matrices*, J. Phys. A: Math. Gen. 31 (1998), 1141–1152.
- [23] Y. Chen, S. M. Manning, *Asymptotic level spacing of the Laguerre ensemble: A Coulomb fluid approach*, J. Phys. A: Math. Gen. 27 (1994), 3615–3620.
- [24] Y. Chen, S. M. Manning, *Distribution of linear statistics in random matrix models (metallic conductance fluctuations)*, J. Phys.: Cond. Matter 6 (1994), 3039.
- [25] Y. Chen, M. R. McKay, *Coulomb fluid, Painlevé transcendents and the information theory of MIMO systems*, IEEE Trans. Info. Theory vol.58, no.7 (2012), 4594–4634.
- [26] Y. Chen, G. Pruessner, *Orthogonal polynomials with discontinuous weights*, J. Phys. A: Math. Gen. 38 (2005), L191–L198.
- [27] Y. Chen, L. Zhang, *Painlevé VI and the unitary Jacobi ensembles*, Stud. Appl. Math. 125 (2010), 91–112.
- [28] P. Deift, *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI.
- [29] P. Deift, A. Its, I. Krasovsky, *Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities*, Anna. Math. 174 (2011), 1243–1299.
- [30] P. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, X. Zhou, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Comm. Pure Appl. Math. 52 (1999), 1491–1552.
- [31] F. J. Dyson, *Statistical theory of the energy levels of complex systems I-III*, J. Math. Phys. 3 (1962), 140–175.
- [32] I. S. Gradshteyn, I. M. Ryzhik, *Table of integrals, series, and products*, seventh ed., Elsevier/Academic Press, Amsterdam, 2007.
- [33] M. Jimbo, *Monodromy problem and the boundary condition for some Painlevé equations*, Publ. RIMS. Kyoto Univ. 18 (1982), 1137–1161.
- [34] I. Krasovsky, *Aspects of Toeplitz determinants*, Prog. in Prob. 64 (2011), 305–324.
- [35] S. Lukyanov, *Finite temperature expectation values of local fields in the sinh-Gordon model*, Nucl. Phys. B 612 (2001), 391–412.
- [36] S. N. Majumdar, G. Schehr, *Top eigenvalue of a random matrix: large deviations and third order phase transition*, J. Stat. Mech.: Theo. Expe. 1 (2014), P01012.
- [37] M. L. Mehta, *Random matrices*, Third edition, San Diego, CA: Elsevier Inc., (2004).

- [38] Y. Ohyama, H. Kawamuko, H. Sakai and K. Okamoto, *Studies the Painlevé equations, V, third Painlevé equations of special type  $P_{III}(D_7)$  and  $P_{III}(D_8)$* , J. Math. Sci. Univ. Tokyo vol.13, no.2 (2006), 145–204.
- [39] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, New York, 1939. American Mathematical Society Colloquium Publications, v. 23.
- [40] C. Texier, S. N. Majumdar, *Wigner time-delay distribution in chaotic cavities and freezing transition*, Phys. Rev. Lett. 110 (2013), 250–602.